Path integral for a relativistic Aharonov-Bohm-Coulomb system

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The path integral for the relativistic spinless Aharonov-Bohm-Coulomb system is solved, and the energy spectra are extracted from the resulting amplitude.

I. INTRODUCTION

With the help of Duru and Kleinert's path-dependent time transformation [1] the list of solvable path integrals has been extended to essentially all potential problems which possess a solvable Schrödinger equation [2,3]. Only recently has the technique been extended to relativistic potential problems [4], followed by two applications [5–8]. Here we'd like to add a further application by solving the path integral of relativistic particle in two dimensions in the presence of an infinitely thin Aharonov-Bohm magnetic field along the z-axis [9] and a $1/r$-Coulomb potential (ABC system). This may be relevant for understanding the behavior of relativistic charged anyons which are restricted to a plane but whose Coulomb field extends into three dimensions [2,10].

II. THE RELATIVISTIC PATH INTEGRAL

Adding a vector potential $\mathbf{A}(\mathbf{x})$ to Kleinert's relativistic path integral for a particle in a potential $V(\mathbf{x})$ [4,2], we find the expression for the fixed-energy amplitude

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc} \int_0^\infty dL \int D\rho \Phi[\rho] \int D^D x \, e^{-A_E/\hbar} \tag{1}$$

with the action

$$A_E = \int_{\tau_a}^{\tau_b} d\tau \left[\frac{M}{2\rho(\tau)} \mathbf{x}'^2(\tau) - i\frac{e}{c}\mathbf{A}(\mathbf{x}) \cdot \mathbf{x}'(\tau) - \rho(\tau)\frac{(E-V)^2}{2Mc^2} + \rho(\tau)\frac{Mc^2}{2} \right]. \tag{2}$$

For the ABC system under consideration, the potential is

$$V(r) = -e^2/r, \tag{3}$$

and the vector potential

$$A_i = 2g\partial_i\theta, \tag{4}$$

where e is the charge and θ is the azimuthal angle around the tube:

$$\theta(\mathbf{x}) = \arctan(x_2/x_1). \tag{5}$$

The associated magnetic field lines are confined to an infinitely thin tube along the z-axis:

$$B_3 = 2g\epsilon_{3jk}\partial_j\partial_k\theta = 2g2\pi\delta^{(2)}(\mathbf{x}_\perp), \tag{6}$$

where $\mathbf{x}_\perp$ is the transverse vector $\mathbf{x}_\perp \equiv (x_1, x_2)$.

Before time-slicing the path integral, we have to regularize it via a so-called f-transformation [2,5], which exchanges the path parameter τ by a new one s:

$$d\tau = ds f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}), \tag{7}$$

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where $f_l(\mathbf{x})$ and $f_r(\mathbf{x})$ are invertible functions whose product is positive. The freedom in choosing $f_{l,r}$ amounts to an invariance under path-dependent-reparametrizations of the path parameter τ in the fixed-energy amplitude (1). By this transformation, the (D+1)-dimensional relativistic fixed-energy amplitude for arbitrary time-independent potential turns into [2,5]

$$
G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[\int d\rho_n \Phi(\rho_n) \right]
$$

$$
\times \frac{f_l(\mathbf{x}_a) f_r(\mathbf{x}_b)}{[2\pi\hbar\epsilon_b^s \rho_b f_l(\mathbf{x}_b) f_r(\mathbf{x}_a)/M]^{D/2}} \prod_{n=1}^{N} \left[\int_{-\infty}^\infty \frac{d^D x_n}{[2\pi\hbar\epsilon_n^s \rho_n f(\mathbf{x}_n)/M]^{D/2}} \right] \exp\left\{ -A^N/\hbar \right\}
\tag{8}
$$

with the s-sliced action

$$
A^N = \sum_{n=1}^{N+1} \left[\frac{M(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1})} - i\frac{e}{c}\mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) \right.
$$

$$
\left. -\epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{(E-V)^2}{2Mc^2} + \epsilon_n^s \rho_n f_l(\mathbf{x}_n) f_r(\mathbf{x}_{n-1}) \frac{Mc^2}{2} \right].
\tag{9}
$$

A family functions which regulates the ABC system is

$$
f_l(\mathbf{x}) = f(\mathbf{x})^{1-\lambda}, \quad f_r(\mathbf{x}) = f(\mathbf{x})^\lambda,
\tag{10}
$$

whose product satisfies $f_l(\mathbf{x})f_r(\mathbf{x}) = f(\mathbf{x}) = r$. Thus arrive at the amplitude

$$
G(\mathbf{x}_b, \mathbf{x}_a; E) \approx \frac{i\hbar}{2Mc} \int_0^\infty dS \prod_{n=1}^{N+1} \left[\int d\rho_n \Phi(\rho_n) \right] \frac{(r_a/r_b)^{1-2\lambda}}{2\pi\hbar\epsilon_b^s \rho_b/M} \prod_{n=2}^{N+1} \left[\int_{-\infty}^\infty \frac{d^2\triangle x_n}{2\pi\hbar\epsilon_n^s \rho_n r_{n-1}/M} \right] \exp\left\{ -\frac{1}{\hbar} A^N \right\},
\tag{11}
$$

with the action

$$
A^N = \sum_{n=1}^{N+1} \left[\frac{M(\mathbf{x}_n - \mathbf{x}_{n-1})^2}{2\epsilon_n^s \rho_n r_n^{1-\lambda} r_{n-1}^\lambda} - i\frac{e}{c}\mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) - \epsilon_n^s \rho_n r_n (r_{n-1}/r_n)^\lambda \frac{(E-V)^2}{2Mc^2} + \epsilon_n^s \rho_n (r_{n-1}/r_n)^\lambda \frac{Mc^2}{2} \right].
\tag{12}
$$

Since the path integral represents the general relativistic resolvent operator, all results must be independent of the splitting parameter λ after going to the continuum limit. Choosing $\lambda = 1/2$, we obtain the continuum limit

$$
A_E[\mathbf{x}, \mathbf{x}'] = \int ds \left[\frac{Mx'^2}{2\rho r} - i\frac{e}{c}\mathbf{A} \cdot \mathbf{x}' - \rho r \frac{(E-V)^2}{2Mc^2} + \rho r \frac{Mc^2}{2} \right].
\tag{13}
$$

We now solve the s-sliced ABC system as in the case of the two-dimensionel Coulomb problem without the Aharonov-Bohm potential [2]. We introducing the *Levi-Cività* transformation

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u^1 & -u^2 \\ u^2 & u^1 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}
\tag{14}
$$

and write this in a matrix form:

$$
\mathbf{x} = A(\mathbf{u})\mathbf{u}.
\tag{15}
$$

For every slice, the coordinate transformation reads

$$
\mathbf{x}_n = A(\mathbf{u}_n)\mathbf{u}_n
\tag{16}
$$

yielding

$$
(\triangle \mathbf{x}_n^i)^2 = 4\bar{\mathbf{u}}_n^2 (\triangle \mathbf{u}_n^i)^2,
\tag{17}
$$

where $\bar{\mathbf{u}}_n \equiv (\mathbf{u}_n + \mathbf{u}_{n-1})/2$. For the sliced AB potential, the *Levi-Cività* transformation yields

$$\mathbf{A}_n \cdot (\mathbf{x}_n - \mathbf{x}_{n-1}) = -2g\frac{(x_2)_n(\triangle x_1)_n - (x_1)_n(\triangle x_2)_n}{r_n^2} = -4g\frac{u_n^2\triangle u_n^1 - u_n^1\triangle u_n^2}{\mathbf{u}_n^2}. \tag{18}$$

Thus we obtain for the path integral (11) the Duru-Kleinert-tranformed expression:

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \frac{i\hbar}{2Mc}\int_0^\infty dS\; e^{SEe^2/\hbar Mc^2}\frac{1}{4}\left[G(\mathbf{u}_b, \mathbf{u}_a; S) + G(-\mathbf{u}_b, \mathbf{u}_a; S)\right], \tag{19}$$

where $G(\mathbf{u}_b, \mathbf{u}_a; S)$ is the s-sliced amplitude of a harmonic oscillator in an Aharonov-Bohm vector potential corresponding to twice the magnetic field (6) in $\mathbf{u}$-space:

$$G(\mathbf{u}_b, \mathbf{u}_a; S) = \prod_{n=1}^{N+1}\left[\int d\rho_n\Phi(\rho_n)\right]\frac{1}{2\pi\hbar\epsilon_b^s\rho_b/M}\prod_{n=1}^N\left[\int_{-\infty}^\infty \frac{d^2 u_n}{2\pi\hbar\epsilon_n^s\rho_n/M}\right]\exp\left\{-\frac{1}{\hbar}A^N\right\},$$

with the action

$$A^N = \sum_{n=1}^{N+1}\left\{\frac{m(\triangle\mathbf{u}_n)^2}{2\epsilon_n^s\rho_n} - 2i\frac{e}{c}(\mathbf{A}_n \cdot \triangle\mathbf{u}_n) + \epsilon_n^s\rho_n\frac{m\omega^2\mathbf{u}_n^2}{2} - \epsilon_n^s\rho_n\frac{\hbar^2 4\alpha^2}{2m\mathbf{u}_n^2}\right\}. \tag{20}$$

Here

$$m = 4M, \qquad \omega^2 = \frac{M^2c^4 - E^2}{4M^2c^2}, \tag{21}$$

and

$$\mathbf{A}_n \cdot \triangle\mathbf{u}_n = -2g\frac{u_n^2\triangle u_n^1 - u_n^1\triangle u_n^2}{\mathbf{u}_n^2}. \tag{22}$$

The symmetrization in $\mathbf{u}_b$ in Eq. (19) is necessary since for each path from $\mathbf{x}_a$ to $\mathbf{x}_b$, there are two paths in the square root space, one from $\mathbf{u}_a$ to $\mathbf{u}_b$ and one from $\mathbf{u}_a$ to $-\mathbf{u}_b$.

As in the two-dimensional Coulomb problem, there are no s-slicing corrections [2].

Let us now analyze the effect come form the magnetic interaction upon the Coulomb system, defining the azimuthal angle $\varphi(\mathbf{u}) = \arctan(u^2/u^1) = \theta(\mathbf{x})/2$ in the u-plane, so that $A_\mu = 2g\partial_\mu\varphi$, $B_3 = 2g\epsilon_{3jk}\partial_j\partial_k\varphi$. Note that derivatives in front of $\varphi(\mathbf{u})$ commute everywhere, except at the origin where Stokes' theorem yields

$$\int d^2 u(\partial_1\partial_2 - \partial_2\partial_1)\varphi = \oint d\varphi = 2\pi \tag{23}$$

The magnetic flux through the tube is defined by the integral

$$\Phi = \int d^2 u B_3. \tag{24}$$

A comparison with the equation for $\varphi(\mathbf{u})$ shows that the coupling constant g is related to the magnetic flux by

$$g = \frac{\Phi}{4\pi}. \tag{25}$$

When inserting $A_\mu = 2g\partial_\mu\varphi$ into Eq. (20), the interaction takes the form

$$A_{\mathrm{mag}} = -2\hbar\mu_0\int_0^S ds\varphi'(s), \tag{26}$$

where $\varphi(s) \equiv \varphi(\mathbf{u}(s))$, and μ_0 is the dimensionless number

$$\mu_0 \equiv -\frac{2eg}{\hbar c}. \tag{27}$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$n = \frac{1}{2\pi} \int_0^S ds \varphi' \tag{28}$$

is the topological invariant with integer values of the winding number n. The magnetic interaction is therefore a purely topological one, its value being

$$A_{\mathrm{mag}} = -\hbar\mu_0 4\pi n. \tag{29}$$

After adding this to the action of Eq. (20) in the radial decomposition of the relativistic path integral [5,6], we rewrite the sum over the azimuthal quantum numbers k via Poisson's summation formula, and obtain

$$G(\mathbf{u}_b, \mathbf{u}_a; S) = \int_{-\infty}^{\infty} d\mu \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_\mu \times \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{i(\mu-2\mu_0)(\varphi_b+2n\pi-\varphi_a)}. \tag{30}$$

Since the winding number n is often not easy to measure experimentally, let us extract observable consequences which are independent of n. The sum over all n forces μ to be equal to $2\mu_0$ modula an arbitrary integer number. the result is

$$G(\mathbf{u}_b, \mathbf{u}_a; S) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{u_b u_a}} G(u_b, u_a; S)_{k+2\mu_0} \frac{1}{2\pi} e^{ik(\varphi_b-\varphi_a)}. \tag{31}$$

We now choose the gauge $\rho(s) = 1$ in Eq. (20). This leads to the Duru-Kleinert transformed action

$$A^N = \int_0^S ds \left[\frac{m\mathbf{u}'^2}{2} - 2i\frac{e}{c}(\mathbf{A} \cdot \mathbf{u}') + \frac{m\omega^2\mathbf{u}^2}{2} - \frac{4\hbar^2\alpha^2}{2m\mathbf{u}^2} \right]. \tag{32}$$

where α denotes the fine-structure constant $\alpha \equiv e^2/\hbar c \approx 1/137$. This action describes a particle of mass $m = 4M$ moving as a function of the "pseudotime" s in an Aharonov-Bohm field and a harmonic oscillator potential of frequency

$$\omega^2 = \frac{Mc^2 - E^2}{4M^2c^2}. \tag{33}$$

In addition, there is an extra attractive potential $V_{\mathrm{extra}} = -4\hbar^2\alpha^2/2m\mathbf{u}^2$ looking like an inverted centrifugal barrier which is comveniently parametrized with the help of a corresonding angular momentum l_{extra}, whose square is negative: $l_{\mathrm{extra}}^2 \equiv -4\alpha^2$, writing $V_{\mathrm{extra}} = \hbar^2 l_{\mathrm{extra}}^2/2m\mathbf{u}^2$. Such an extra potential can easily be incorporated into the amplitude of the pure Coulomb system by a technique developed in the treatment of the radial part of the harmonic oscillator path integral [14], yielding a radial amplitude for the azimuthal quantum number k:

$$G(u_b, u_a; S)_k = \frac{m}{\hbar} \frac{\omega\sqrt{u_b u_a}}{\sinh\omega s} e^{-\frac{m\omega}{2\hbar}(u_b^2+u_a^2)\coth\omega s} I_{\sqrt{|k|^2-4\alpha^2}}\left(\frac{m}{\hbar}\frac{\omega u_b u_a}{\sinh\omega s}\right). \tag{34}$$

where I_ν is the modified Bessel function. Incorporating also the effect of the Aharonov-Bohm potential yields

$$G(u_b, u_a; S)_{k+2\mu_0} = \frac{m}{\hbar} \frac{\omega\sqrt{u_b u_a}}{\sinh\omega s} e^{-\frac{m\omega}{2\hbar}(u_b^2+u_a^2)\coth\omega s} I_{\sqrt{|k+2\mu_0|^2-4\alpha^2}}\left(\frac{m}{\hbar}\frac{\omega u_b u_a}{\sinh\omega s}\right). \tag{35}$$

These radial amplitudes can now be combined with angular wave functions to find the full amplitude (30).

Inserting the result into the integral representation (19) for the resolvent, we use polar coordinates in $\mathbf{x}$-space with $\theta = 2\varphi, r = u^2$, and obtain the expression

$$G(\mathbf{x}_b, \mathbf{x}_a; E) = \sum_{k=-\infty}^{\infty} G(r_b, r_a; E)_k \frac{1}{2\pi} e^{ik(\theta_b-\theta_a)}, \tag{36}$$

where

$$G(r_b, r_a; E)_k = \frac{i\hbar}{2Mc} \frac{2M}{\hbar} \int_0^{\infty} dS\ e^{e^2 ES/\hbar Mc^2} \frac{\omega}{\sinh\omega s} e^{-\frac{m\omega}{2\hbar}(r_b+r_a)\coth\omega s} I_{\sqrt{4|k+\mu_0|^2-4\alpha^2}}\left(\frac{m}{\hbar}\frac{\omega\sqrt{r_b r_a}}{\sinh\omega s}\right). \tag{37}$$

The integral can be calculated with the help of the formula

$$\int_0^\infty dy \frac{e^{2\nu y}}{\sinh y} \exp\left[-\frac{t}{2}\left(\zeta_a + \zeta_b\right)\coth y\right] I_\mu\left(\frac{t\sqrt{\zeta_b\zeta_a}}{\sinh y}\right) = \frac{\Gamma\left((1+\mu)/2 - \nu\right)}{t\sqrt{\zeta_b\zeta_a}\Gamma\left(\mu+1\right)} W_{\nu,\mu/2}\left(t\zeta_b\right) M_{\nu,\mu/2}\left(t\zeta_b\right), \qquad (38)$$

with the range of validity

$$\zeta_b > \zeta_a > 0,$$
$$Re[(1+\mu)/2 - \nu] > 0,$$
$$Re(t) > 0, |\arg t| < \pi,$$

where $M_{\mu,\nu}$ and $W_{\mu,\nu}$ are the Whittaker functions. In this way, we obtain the final result for the radial amplitude valid for $u_b > u_a$:

$$G(r_b, r_a; E)_k = \frac{i\hbar}{2Mc} \frac{4Mc}{\sqrt{M^2c^4 - E^2}} \times \frac{\Gamma\left(1/2 + \sqrt{|k+\mu_0|^2 - \alpha^2} - E\alpha/\sqrt{M^2c^4 - E^2}\right)}{\sqrt{r_a r_b}\Gamma\left(2\sqrt{|k+\mu_0|^2 - \alpha^2} + 1\right)}$$

$$\times W_{E\alpha/\sqrt{M^2c^4 - E^2},\sqrt{|k+\mu_0|^2 - \alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}r_b\right) M_{E\alpha/\sqrt{M^2c^4 - E^2},\sqrt{|k+\mu_0|^2 - \alpha^2}}\left(\frac{2}{\hbar c}\sqrt{M^2c^4 - E^2}r_a\right). \quad (39)$$

The energy spectra can be extracted from the poles. They are determined by

$$\frac{1}{2} + \sqrt{|k+\mu_0|^2 - \alpha^2} - \frac{E\alpha}{\sqrt{M^2c^4 - E^2}} = -n_r, \qquad n_r = 0, 1, 2, \cdots. \qquad (40)$$

Expanding this equation into powers of α, we get

$$E_{nk} = \pm Mc^2 \left\{ 1 - \frac{1}{2}\left[\frac{\alpha}{n + |k+\mu_0| - 1/2}\right]^2 - \frac{\alpha^4}{(n + |k+\mu_0| - 1/2)^3}\left[\frac{1}{2|k+\mu_0|} - \frac{3}{8(n+|k+\mu_0|-1/2)}\right] + \cdots \right\}, \quad (41)$$

for $n = 1, 2, 3, \cdots$. In the non-relativistic limit, the spectra reduces to that in Ref. [11–13].

The alert reader will have noted the similarity of the techniques used in this paper to those leading to the solution of the path integral of the dionium atom [14].

[1] H. Duru and H. Kleinert, Phys. Letters **B84**, 185 (1979); Fortschr. d. Phys. **30**, 401 (1982)

[2] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, World Scientific, Singapore, 1995 (sec. ed.) and references therein.

[3] A. Inomata, H. Kuratsuji, and C. C. Gerry, *Path Integrals and Coherent States of SU(2) and SU(1,1)*, World Scientific, Singapore, 1992 and references therein.

[4] H. Kleinert, Phys. Lett. **A 212**, 15 (1996).

[5] D. H. Lin, J. Phys. **A 30**, 3201 (1997).

[6] D. H. Lin, J. Phys. **A 30**, 4365 (1997).

[7] B. Bentag, L. Chetouani, L. Guechi, and T. F. Hammann, Nuovo Cimento **B 111**, 99 (1996).

[8] The work of R. Ho and A. Inomata, Phys. Rev. Lett. **48**, 231 (1982), although dealing with a relativistic particle problem, is of a different kind since it does not start from a sum over relativistic paths but from a relativistic Schrödinger equation.

[9] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[10] M. Peskin and A. Tonomura, *The Aharonov-Bohm Effect*, Springer, New York, 1988.

[11] C. R. Hagen, Phys. Rev. **D 48**, 5936 (1993).

[12] L. Chetouani, L. Guechi and T. F. Hammann, J. Math. Phys. **30**, 655 (1989).

[13] D. K. Park and S. K. Yoo, hep-th/9707024.

[14] See Sections 8.6 and 16.6 in [2]; and Phys. Lett. **A 116**, 201 (1986).